Computer Graphics

Chapter_4

2D Transformations

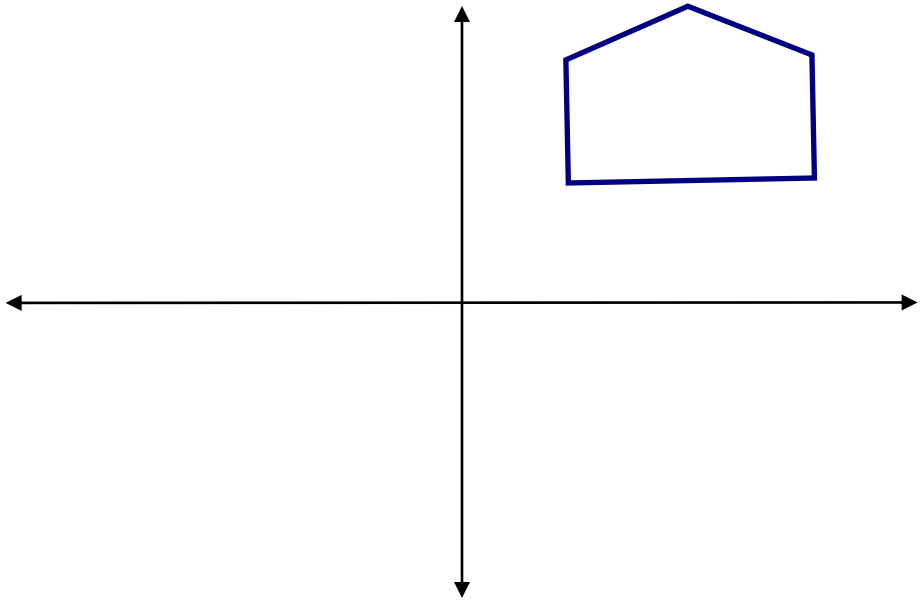
Lecture #4

General Pivot-Point Rotation

- Rotate an object with the vertices $V_1,\ V_2, V_3, V_4, \& V_5$ using a rotation angle, $\boldsymbol{\theta}$, in the counter-clockwise direction about its centroid.

Solution

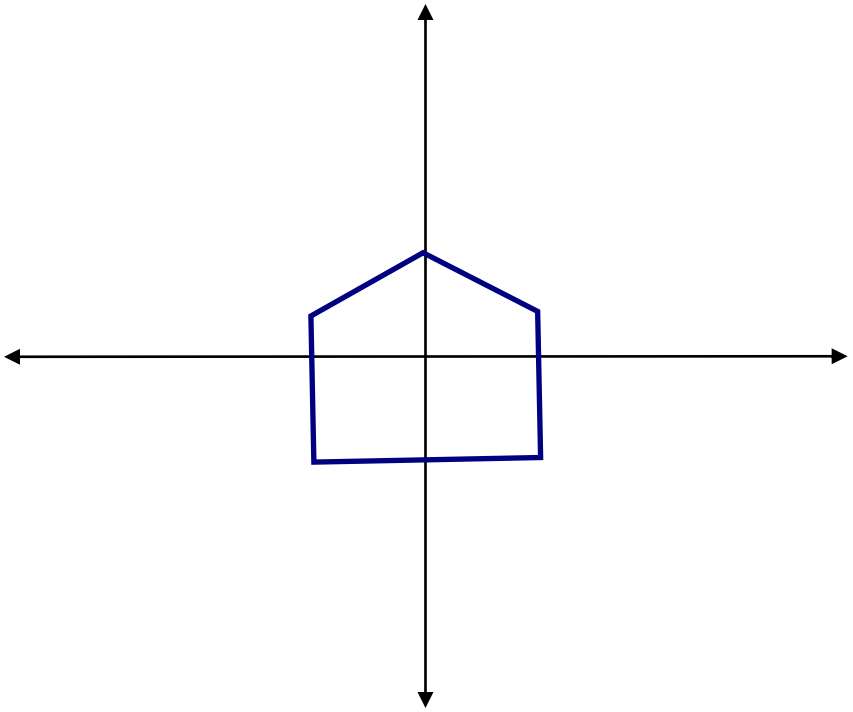
Original Object



Step 1:

Translate the object where its centroid, (x_c, y_c), coincides with the coordinate origin.

Translated Object

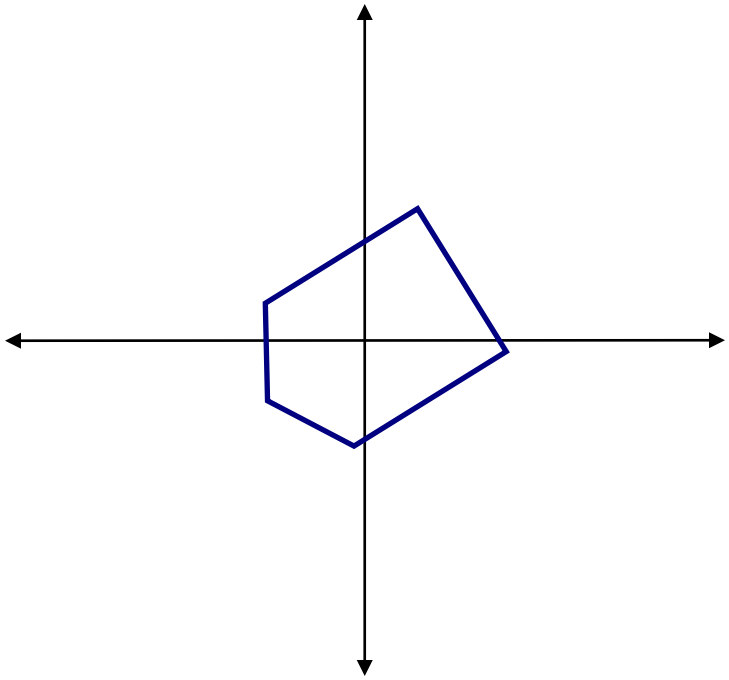


$$A^1 = T(-x_c, -y_c)\,\mathbf{A}$$

Step 2:

Rotate the object with the ,$\boldsymbol{\theta}$, in the counter-clockwise direction about the rotation axis.

Rotated Object

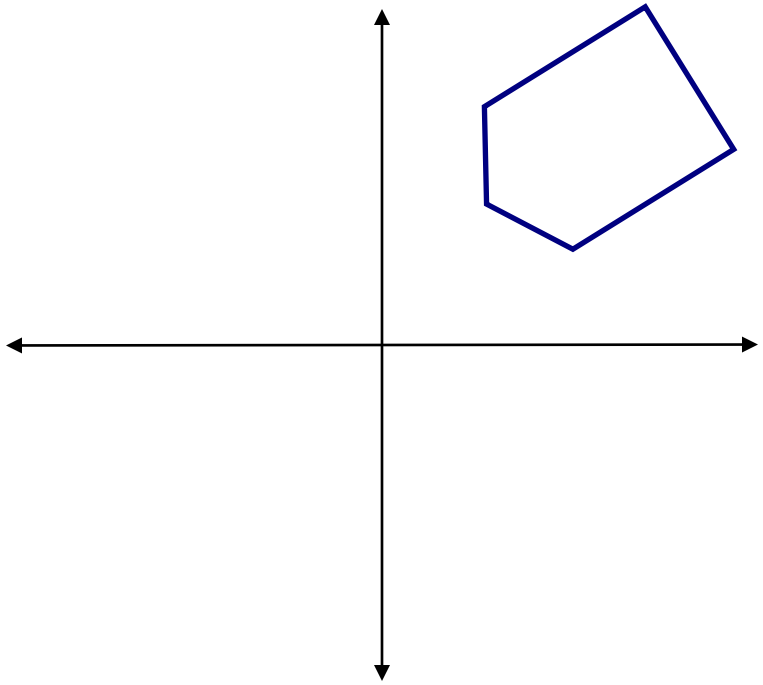


$$\boldsymbol{A^2 = R(\theta)\ A^1}$$

Step 3:

Translate the object to the original position of its centroid.

Translated Object



$$A^3 = T(x_c, y_c)\, A^2$$

The final transformation is the result of the implementation of the three transformations.

REMEMBER: Matrix multiplication is not commutative. So, order matters.

Since: $$A^1 = T(-x_c, -y_c)\ \mathbf{A}$$

Then: $$A^2 = R(\theta)\ A^1$$

$$A^2 = R(\theta)\ T(-x_c, -y_c)\ \mathbf{A}$$

Finally:

$$A^3 = T(x_c, y_c)\ A^2$$

$$A^3 = T(x_c, y_c) R(\theta)\ T(-x_c, -y_c)\ \mathbf{A}$$

Use the matrix from, we could write:

$$A' = \begin{bmatrix} 1 & 0 & x_c \\ 0 & 1 & y_c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & \mathbf{0} \\ \sin\theta & \cos\theta & \mathbf{0} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ 0 & 0 & 1 \end{bmatrix} A$$

$$A' = \begin{bmatrix} \cos\theta & -\sin\theta & x_c \\ \sin\theta & \cos\theta & y_c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ 0 & 0 & 1 \end{bmatrix} A$$

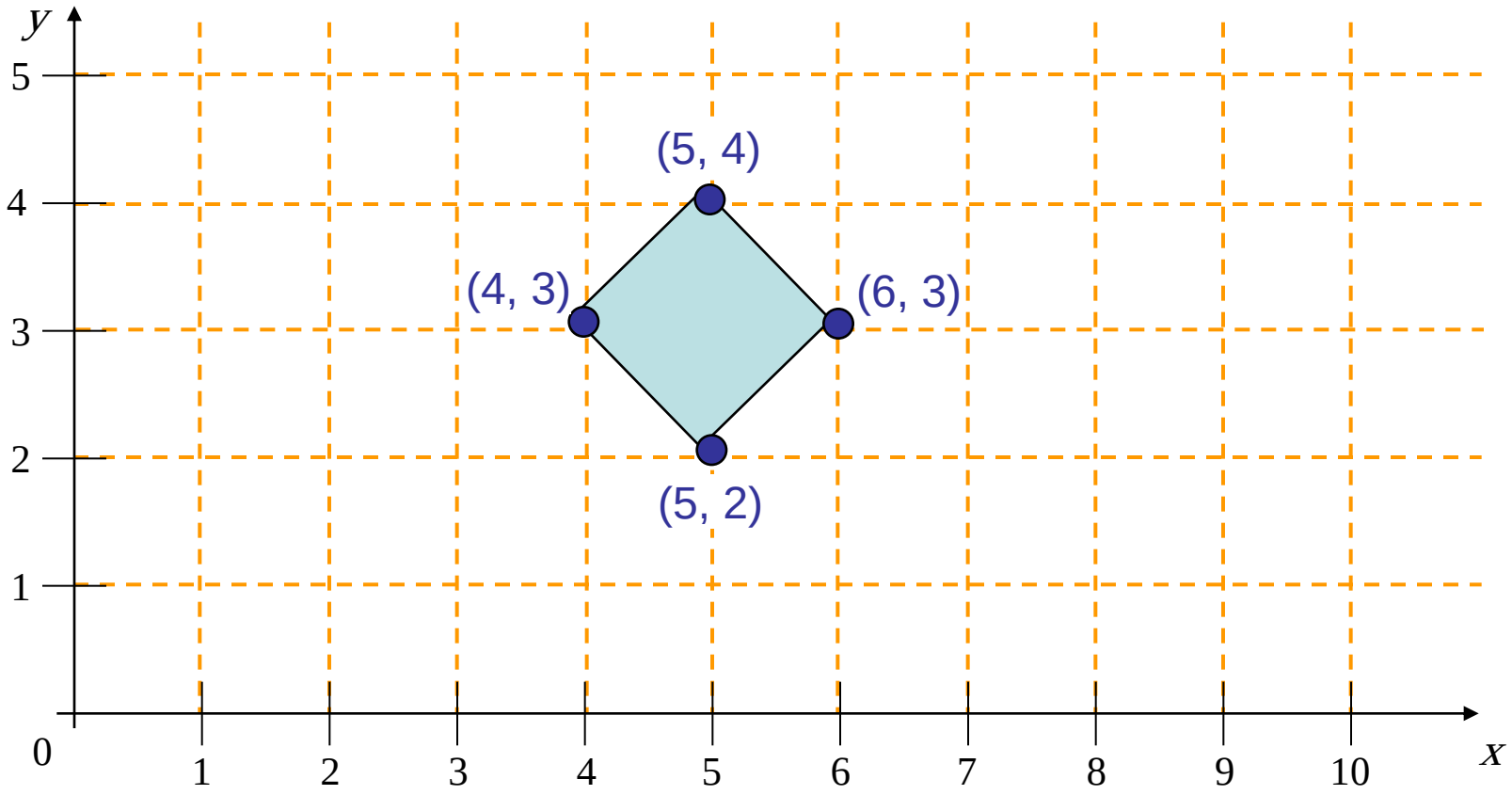
$$A' = \begin{bmatrix} \cos\theta & -\sin\theta & -x_c\cos\theta + y_c\sin\theta + x_c \\ \sin\theta & \cos\theta & -x_c\sin\theta - y_c\cos\theta + y_c \\ 0 & 0 & 1 \end{bmatrix} A$$

Recall matrix multiplication, then:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & x_c(\mathbf{1} - \cos\theta) + y_c\sin\theta \\ \sin\theta & \cos\theta & y_c(\mathbf{1} - \cos\theta) - x_c\sin\theta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Exercises

Use matrix multiplication to calculate the rotation of the shape below by 45° about its center (5, 3)

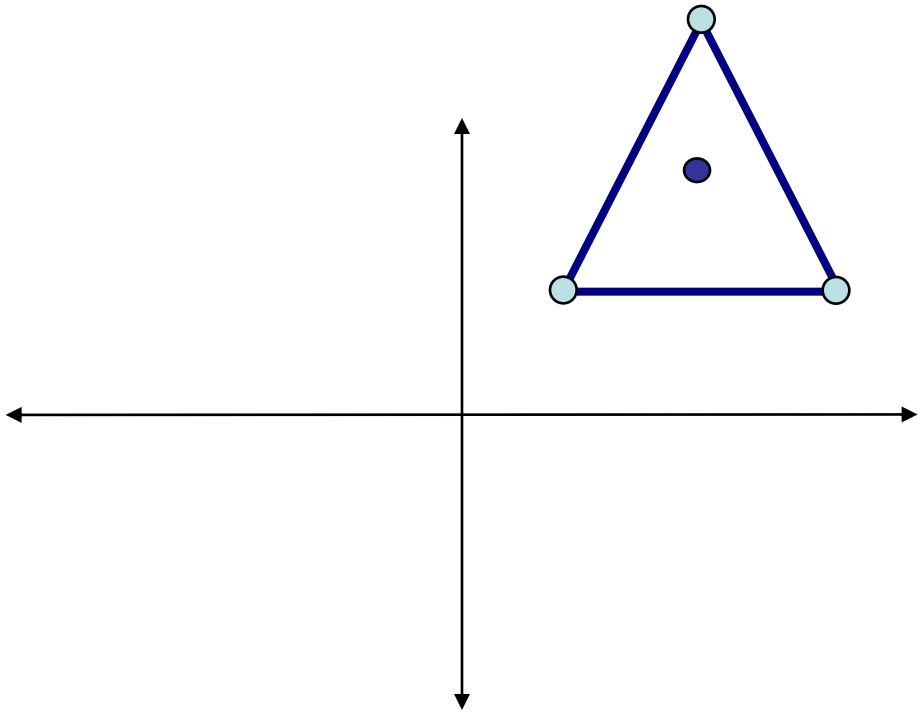


General Fixed-Point Scaling

- Scale an object about a general fixed point (x_g, y_g).

Solution

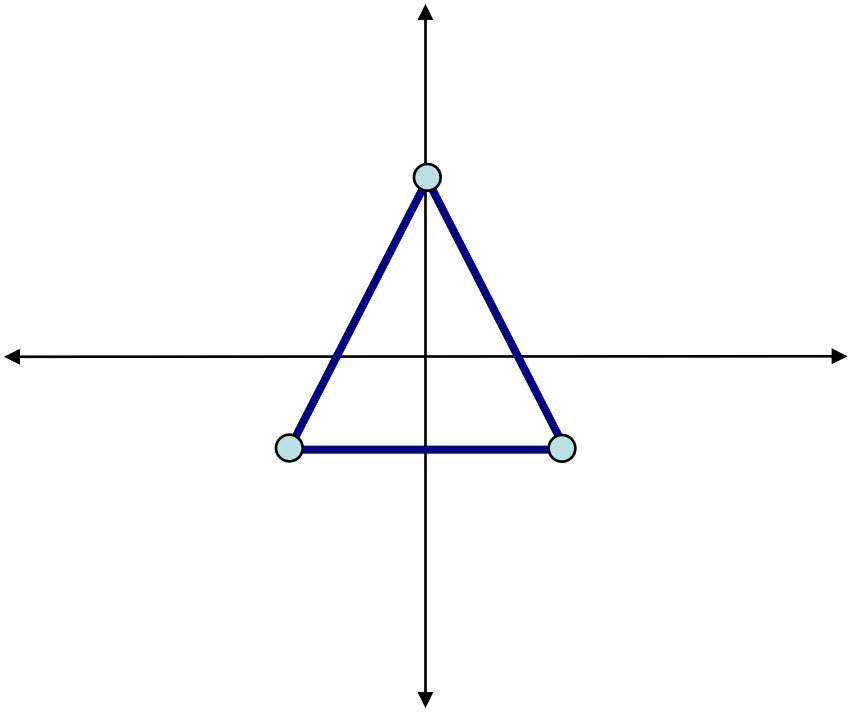
Original Object



Step 1:

Translate the object where the general point, (x_g, y_g), coincides with the coordinate origin.

Translated Object

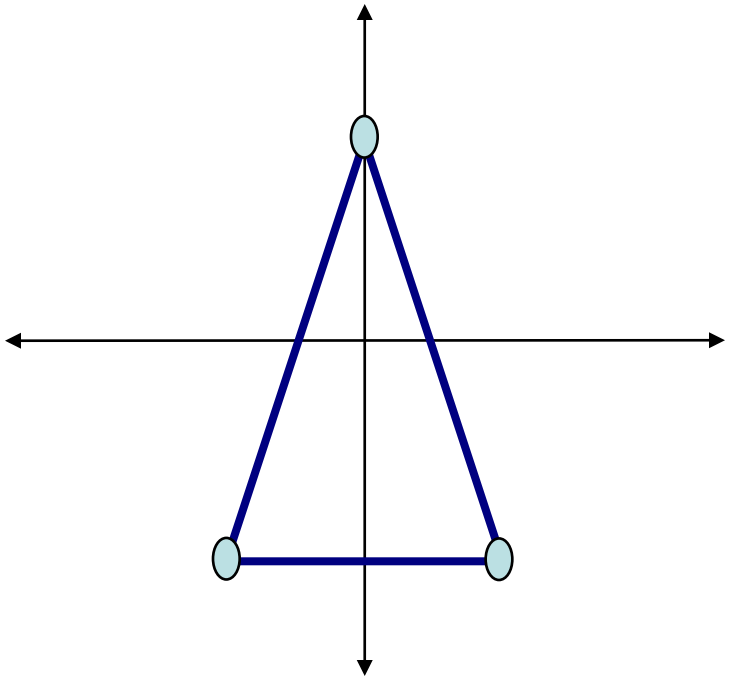


$$A^1 = T(-x_g, -y_g)\,A$$

Step 2:

Scale the object with the scaling factors (s_x, s_y) about the coordinate origin.

Scaled Object

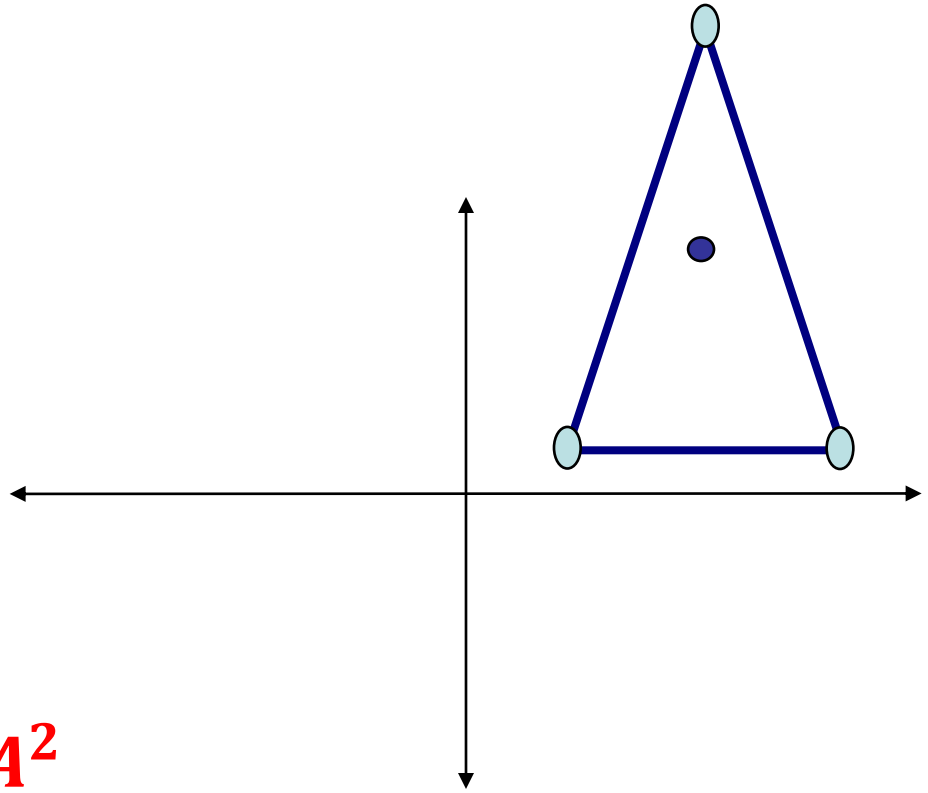


$$A^2 = S(s_x, s_y)\ A^1$$

Step 3:

Translate the object to the original position.

Translated Object



$$A^3 = T(x_g, y_g)\, A^2$$

The final transformation is the result of the implementation of the three transformations.

REMEMBER: Matrix multiplication is not commutative. So, order matters.

Since: $$A^1 = T(-x_g, -y_g)\ \mathbf{A}$$

Then: $$A^2 = S(s_x, s_y)\ A^1$$

$$A^2 = S(s_x, s_y)T(-x_g, -y_g)\ \mathbf{A}$$

Finally:

$$A^3 = T(x_g, y_g)\ A^2$$

$$A^3 = T(x_g, y_g)S(s_x, s_y)T(-x_g, -y_g)\ \mathbf{A}$$

Use the matrix from, we could write:

$$A^{/} = \begin{bmatrix} 1 & 0 & x_g \\ 0 & 1 & y_g \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & \mathbf{0} \\ 0 & s_y & \mathbf{0} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_g \\ 0 & 1 & -y_g \\ 0 & 0 & 1 \end{bmatrix} A$$

$$A^{/} = \begin{bmatrix} s_x & 0 & x_g \\ 0 & s_y & y_g \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_g \\ 0 & 1 & -y_g \\ 0 & 0 & 1 \end{bmatrix} A$$

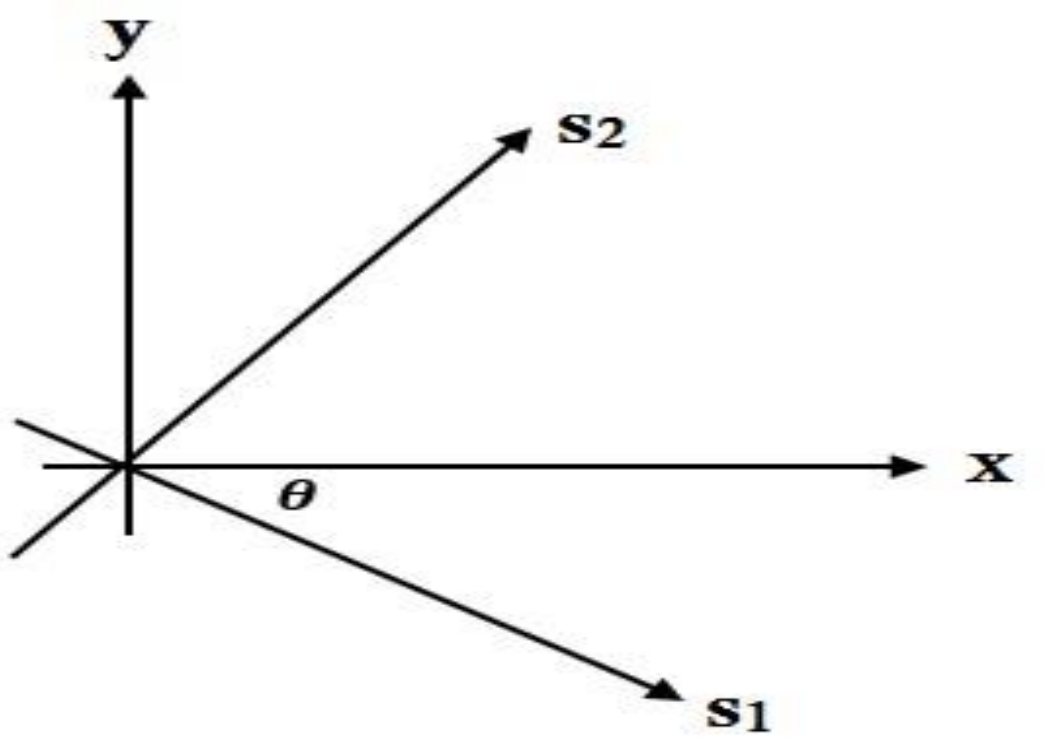
$$A^{/} = \begin{bmatrix} s_x & 0 & -x_g s_x + x_g \\ 0 & s_y & -y_g s_{y+} y_g \\ 0 & 0 & 1 \end{bmatrix} A$$

Recall matrix multiplication, then:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & x_g\,(\mathbf{1} - s_x) \\ 0 & s_y & y_g(\mathbf{1} - s_y) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

General Scaling Directions

Suppose we want to apply scaling factors with values specified by scaling parameters s_x and s_y in the directions shown in the following figure.



Step (1):

We perform a rotation so that the directions for $\boldsymbol{s_x}$ and $\boldsymbol{s_y}$ coincide with the $\boldsymbol{x-}$ and $\boldsymbol{y-}$ axes, respectively.

This means a counterclockwise rotation with an angle $\boldsymbol{\theta}$.

$$\boldsymbol{A^1 = R(\theta)\, A}$$

Step (2):

Now, the scaling transformation using s_x and s_y is applied.

$$A^2 = S(s_x, s_y)\, A^1$$

Step (3):

Perform opposite rotation to return the object points to their original orientations

$$A^3 = R(-\theta)A^2$$

The final transformation is the result of the implementation of the three transformations.

Again, remember: Matrix multiplication is not commutative. So, order matters.

Since:

$$A^1 = R(\theta)\ A$$

Then:

$$A^2 = S(s_x\ , s_y)\ A^1$$

$$A^2 = S(s_x\ , s_y)\ R(\theta)\ \mathbf{A}$$

Finally:

$$A^3 = R(-\theta)\ A^2$$

$$A^3 = R(-\theta)\ S(s_x\ , s_y)\ R(\theta)\ \mathbf{A}$$

Use the matrix from, we could write:

$$A' = \begin{bmatrix} \cos\theta & \sin\theta & \mathbf{0} \\ -\sin\theta & \cos\theta & \mathbf{0} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & \mathbf{0} \\ 0 & s_y & \mathbf{0} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & \mathbf{0} \\ \sin\theta & \cos\theta & \mathbf{0} \\ 0 & 0 & 1 \end{bmatrix} A$$

Recall matrix multiplication, then:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x\cos^2\theta + s_y\sin^2\theta & (s_y - s_x)\cos\boldsymbol{\theta}\sin\boldsymbol{\theta} & 0 \\ (s_y - s_x)\cos\boldsymbol{\theta}\sin\boldsymbol{\theta} & s_x\sin^2\theta + s_y\cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Example:

Stretching the square with vertices $V_1(0,0)$, $V_2(1,0)$, $V_3(1,1)$ & $V_4(0,1)$ along the diagonal from $(0,0)$ to $(1,1)$ using the scaling parameters $s_x = 1$, $s_y = 2$.

Solution:

- The diagonal from $(0,0)$ to $(1,1)$ is a straight line which make an angle $\mathbf{45^0}$.
- We rotate the diagonal onto they $y-$axis, so, both coordinate systems are coincides.

- Substitute s_x, s_y and θ by 1, 2 and $\mathbf{45^0}$ respectively. Then, the transformation matrix will be rewritten as follows:

$$\begin{bmatrix} 1 * \cos^2 45^0 + 2 * \sin^2 45^0 & (2-1)\cos 45^0 \sin 45^0 & 0 \\ (2-1)\cos 45^0 \sin 45^0 & 1 * \cos^2 45^0 + 2 * \sin^2 45^0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 * \frac{1}{2} + 2 * \frac{1}{2} & (1) * \frac{1}{2} & 0 \\ (1) * \frac{1}{2} & 1 * \frac{1}{2} + 2 * \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Vertex, V_1:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Vertex, V_2:

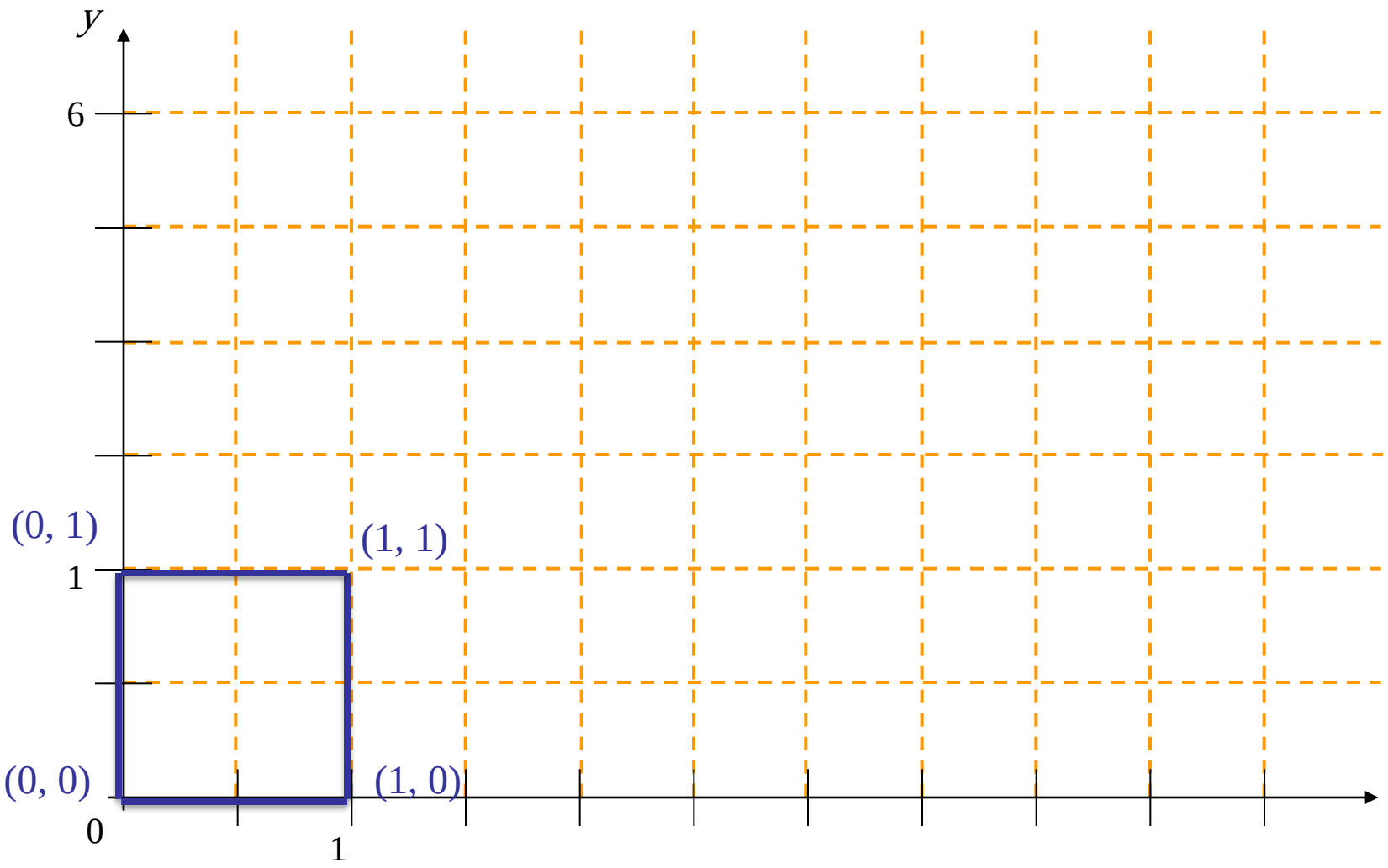
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Vertex, V_3:

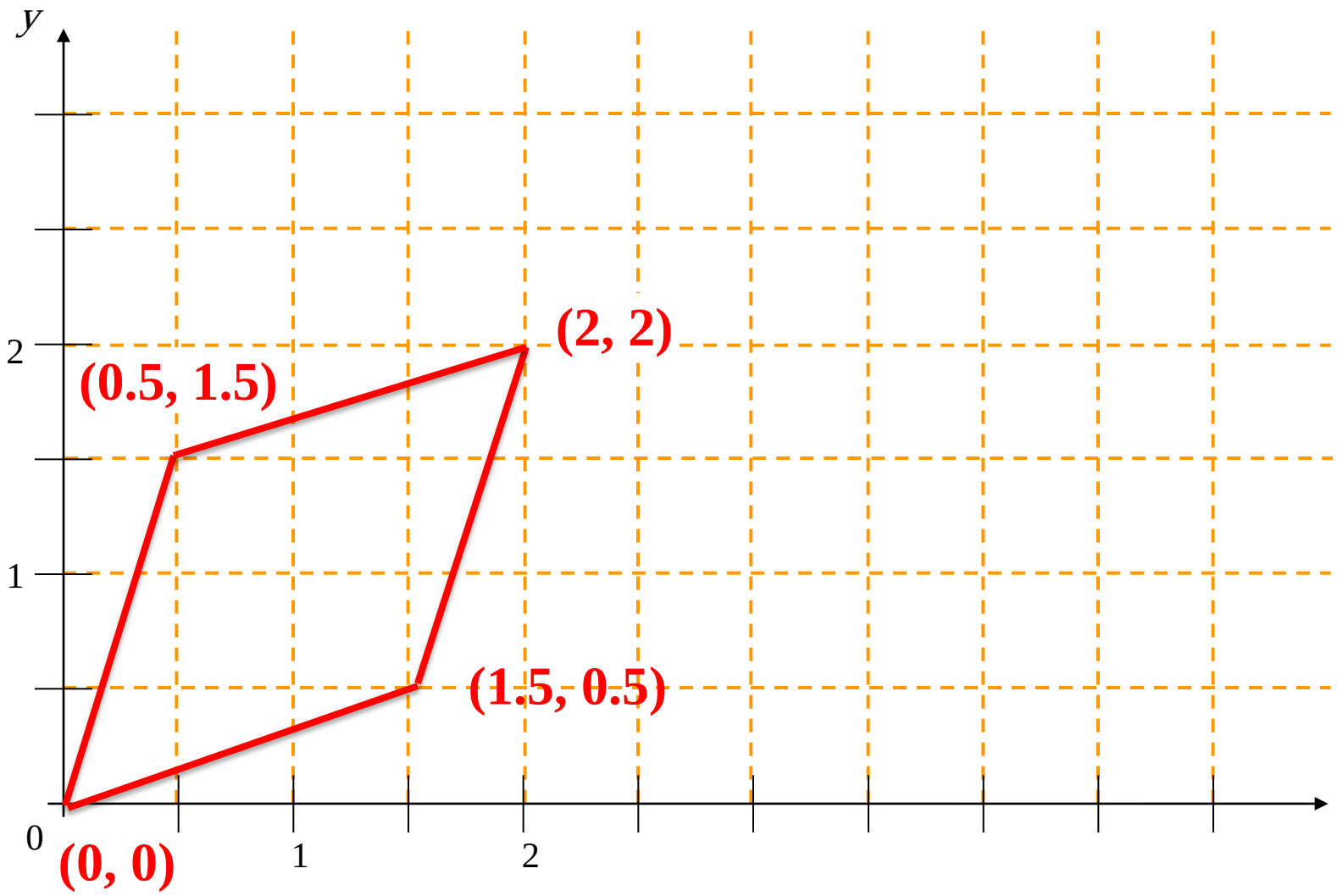
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{2} \\ \frac{4}{2} \\ 1 \end{bmatrix}$$

Vertex, V_4:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix}$$



Original Object



Scaled Object